Learning PID Structures in an Introductory Course of Automatic Control

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***Abstract*—Proportional-integral-derivative (PID) controllers are described in most automatic control textbooks. The application of PID controllers is widely spread in automation of mechanical processes where control of motors is of concern. This paper focuses on implementation of the PID control when used for regulation of dc motors. Two basic PID structures for position regulation of armature-controlled dc motors are studied: the classical structure based on PI position loop plus velocity feedback, and a hierarchical two-loop feedback structure invoking a velocity proportional-integral (PI) inner loop. It is shown that the latter requires simpler stability conditions than the former. Basic concepts from automatic control are evoked in this study, namely, transfer function, characteristic polynomial, stability, and Routh–Hurwitz criterion. Experiments on a direct-drive motor are provided to illustrate the PID control performance.**

***Index Terms*—DC motor control, PID control, Routh–Hurwitz criterion, stability.**

I. Introduction

ONE of the most useful control algorithms in linear and nonlinear control systems is proportional-integral-derivative (PID) control. PID control for position regulation of dc motors is a popular basic example evoked in many linear control textbooks [1], [2]. Notwithstanding, the PID control of dc motors can lead to an unstable closed-loop system as long as the PID gains are unsuitably selected.

Depending on the signals available for measurement, the PID control can be implemented evoking several structures [3], [4]. This paper studies—from a stability viewpoint—two structures for implementing the PID control of dc motors assuming that shaft position and velocity are available for measurement. The first structure arises from a proportional-integral (PI) position loop plus velocity feedback; the second one derives from a hierarchical structure based in a velocity inner loop plus a position outer loop.

In this paper, the authors show that implementation of the PID control based on the philosophy of two loops for constructing a hierarchical controller keeps the closed-loop system stability with conditions less stringent than those required when the PID control is implemented as a PI position loop plus velocity feedback.

A classical linear description of an armature-controlled dc motor—neglecting armature inductance—is given by [1], [2], [5], and [6]

$$J\ddot{q} + \left[f_v + \frac{K_a K_b}{R_a}\right]\dot{q} = \frac{K_a}{R_a} v \tag{1}$$

where

- q output shaft angular position;
- v input voltage;
- J rotor inertia;
- f_v viscous friction.

The constants K_a, K_b, and R_a are electrical characteristics of the motor. All these parameters are strictly positive constants.

The position regulation aim consists of ensuring

$$\lim_{t\to\infty} q(t) = q_d \tag{2}$$

where q_d is a constant which specifies the dc motor desired shaft angular position.

The basic textbook structure of the PID control law driven by the shaft position error defined as $\tilde{q} = q_d - q$ is given by

$$\begin{aligned} v &= \left[k_p' + \frac{k_i'}{p} + k_v' p\right]\tilde{q} \\ &= k_p'\tilde{q} + k_i'\int_0^t \tilde{q}(\sigma)\,d\sigma + k_v'\dot{\tilde{q}} \end{aligned} \tag{3}$$

where $p = (d/dt)$ is the differential operatorand k_p', k_i', and k_v' are the proportional, integral, and derivative gains, respectively.

II. PID Control: Implementation Based on PI Feedback of Position Error

PID control (3) can be implemented as depicted in the block diagram of Fig. 1. This implementation corresponds to a PI feedback of position error $\tilde{q}$ plus velocity feedback according to

$$v = k_p'\tilde{q} + k_i'\xi - k_v'\dot{q} \tag{4}$$

$$\dot{\xi} = \tilde{q}. \tag{5}$$

The closed-loop equation is obtained by substituting the control law (4) into the motor model (1)

$$J\ddot{\tilde{q}} + \left[f_v + \frac{K_a K_b}{R_a} + \frac{K_a k_v'}{R_a}\right]\dot{\tilde{q}} + \frac{K_a k_p'}{R_a}\tilde{q} + \frac{K_a k_i'}{R_a}\xi = 0 \tag{6}$$

where ξ is defined in (5). The third-order characteristic polynomial associated with (6) is

$$s^3 + \frac{1}{J}\left[f_v + \frac{K_a K_b}{R_a} + \frac{K_a k_v'}{R_a}\right]s^2 + \frac{K_a k_p'}{JR_a}s + \frac{K_a k_i'}{JR_a} \tag{7}$$

where s is the Laplace complex variable.

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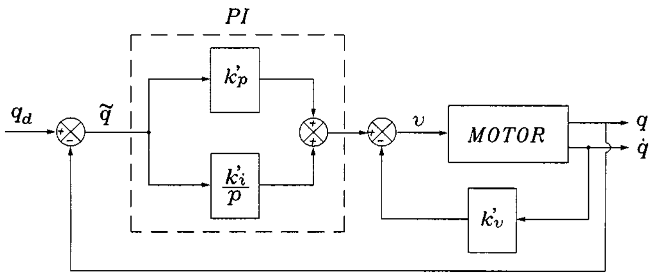

Fig. 1. PID control based on PI feedback of position error $\tilde{q}$.

A sufficient condition for a feedback system to be stable[1] is that all poles of the system transfer function have negative real parts [2], [7]. Using Routh–Hurwitz stability criterion [2], [7], one can obtain the following simple condition for the characteristic polynomial (7) to be stable[2]

$$\left[f_v + \frac{K_a K_b}{R_a} + \frac{K_a k'_v}{R_a}\right] k'_p - J k'_i > 0. \tag{8}$$

According to (8), one should select carefully the gains k'_p, k'_i, and k'_v. For example, once k'_v and k'_p (>0) are chosen, then the integral gain k'_i should be adjusted to satisfy with (8). However, this action requires the knowledge—or suitable bounds—of all the motor parameters: f_v, K_a, K_b, R_a, and J.

III. PID Control: Implementation Based on PI Feedback of Velocity Error

The PID control (3) for position regulation of motors can also be implemented as a control scheme based on two loops as depicted in Fig. 2.

First, one must consider *velocity control* using the following PI controller which defines the inner loop control

$$v = \overline{J}\dot{\omega}_d + k_v\tilde{\omega} + k_i z \tag{9}$$

$$\dot{z} = \tilde{\omega} \tag{10}$$

where ω_d stands for the shaft velocity command; $\tilde{\omega}$ denotes the inner loop velocity error defined by

$$\tilde{\omega} = \omega_d - \dot{q} \tag{11}$$

and the control gains k_v, k_i, and $\overline{J}$ are assumed positive constants. One should note that the controller (9) and (10) has an inverse-dynamics structure with PI velocity error feedback plus acceleration feedforward where the constant $\overline{J}$ is rendered as an *estimation* of the product $(R_a/K_a)\,J$.

The outer loop control is achieved defining the velocity command ω_d as

$$\omega_d = k\tilde{q} \tag{12}$$

where k is a positive constant. Because q_d is assumed to be constant, then the outer loop velocity error defined by $\dot{\tilde{q}} = \dot{q}_d - \dot{q}$ becomes $\dot{\tilde{q}} = -\dot{q}$.

[1]A linear time-invariant system is stable if its output is bounded for any bounded input [2].

[2]See the Appendix for the use of Routh–Hurwitz criterion in a third-order system.

The control action (9) can be expressed in terms of the position error $\tilde{q}$ using (11) and (12) as

$$v = [\overline{J}k + k_v]\dot{\tilde{q}} + kk_i\int_0^t \tilde{q}(\sigma)\,d\sigma + [kk_v + k_i]\tilde{q} \tag{13}$$

which has the structure of the PID control (3). The relationship between the gains of these controllers is

$$k'_p = kk_v + k_i, \tag{14}$$

$$k'_i = kk_i, \quad \text{and} \tag{15}$$

$$k'_v = \overline{J}k + k_v. \tag{16}$$

Substituting the control law (13) into the motor equation (1) leads to the closed-loop system in terms of the position error $\tilde{q}$

$$J\ddot{\tilde{q}} + \left[\frac{K_a}{R_a}\left[\overline{J}k + k_v\right] + f_d\right]\dot{\tilde{q}} + \frac{K_a}{R_a}[kk_v + k_i]\tilde{q} + \frac{K_a}{R_a}kk_i\underbrace{\int_0^t \tilde{q}(\sigma)\,d\sigma}_{\xi} = 0 \tag{17}$$

where f_d is defined by

$$f_d = f_v + \frac{K_a K_b}{R_a}.$$

The third-order characteristic polynomial of system (17) is given by

$$s^3 + \left[\frac{K_a}{R_a}\frac{1}{J}k_v + \frac{f_d}{J} + \frac{K_a}{R_a}\frac{\overline{J}}{J}k\right]s^2 + \left[\frac{K_a}{R_a}\frac{1}{J}kk_v + \frac{K_a}{R_a}\frac{1}{J}k_i\right]s + \frac{K_a}{R_a}\frac{1}{J}kk_i. \tag{18}$$

One can again use the Routh–Hurwitz criterion to find conditions on which the polynomial (18) has zeros with negative real part. A sufficient condition for polynomial (18) to have zeros with negative real part[3] is that its coefficients be positive, i.e.,

$$\frac{K_a}{R_a}\frac{1}{J}k_v + \frac{f_d}{J} + \frac{K_a}{R_a}\frac{\overline{J}}{J}k > 0 \tag{19}$$

$$\frac{K_a}{R_a}\frac{1}{J}kk_v + \frac{K_a}{R_a}\frac{1}{J}k_i > 0 \tag{20}$$

$$\frac{K_a}{R_a}\frac{1}{J}kk_i > 0 \tag{21}$$

and

$$\left[\frac{K_a}{R_a}\frac{1}{J}k_v + \frac{f_d}{J} + \frac{K_a}{R_a}\frac{\overline{J}}{J}k\right]kk_v + \left[\frac{K_a}{R_a}\frac{1}{J}k_v + \frac{f_d}{J}\right]k_i + \left[\frac{\overline{J}}{J}\frac{K_a}{R_a} - 1\right]kk_i > 0 \tag{22}$$

be satisfied. So, in order to satisfy conditions (19)–(22) for ensuring stability of the closed-loop system, it is enough that the gains k, k_v, k_i be positive and the parameter $\overline{J}$ be chosen such that

$$\overline{J} > \frac{R_a}{K_a}J. \tag{23}$$

[3]See the Appendix.

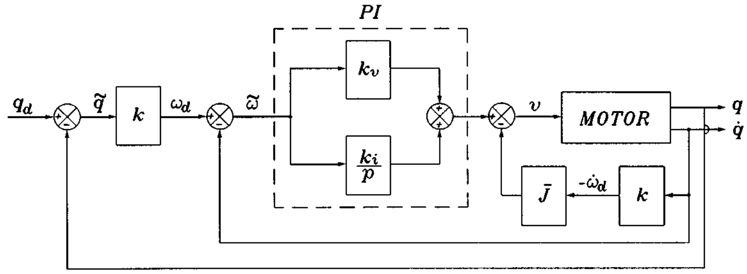

Fig. 2. PID control based on PI feedback of velocity error $\tilde{\omega}$.

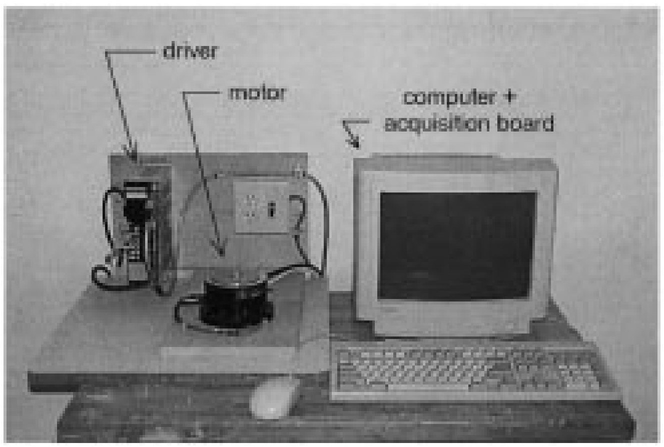

Fig. 3. Experimental setup.

In sum, the implementation of the PID controller as two-loop of feedback (9), (10), and (12) ensures closed-loop stability for any selection of the controller parameters k, k_v, and k_i provided that an upper bound $\overline{J}$ on the product $(R_a/K_a)\,J$ is available.

It is worth noticing that condition (23) is easier to check than (8) evoked for stability of the PID implementation (4) and (5) which needs in addition to the controller gains, also knowledge of the following motor parameters: f_v, and K_b.

IV. Experimental Results

Experiments on a direct-drive motor have been carried out in order to show the performance of the PID control.

The experimental setup is depicted in Fig. 3. The motor used in the experiments is the model DM1004C from Compumotor. This motor is equipped with an optical incremental encoder which provides a resolution of 655 360 pulses per revolution.

The control algorithm based on PI velocity feedback (9)–(12) was coded in C language and executed at $h = 2.5$ ms sampling interval in a PC equipped with a data acquisition board MFIO-3A from Precision MicroDynamics.

Experiments showed that static and Coulomb friction effects at the motor shaft were present. These experiments are described in details in [8]. Since they depended in a complex manner on the motor position and velocity, the authors decide to consider them as disturbances during experiments. The motor model has the structure (1) where numerical value of the parameters are listed in Table I.

TABLE I
PARAMETERS OF THE MOTOR

parameter	*value*	*units*
J	0.0025	kg m^2/rad
$f_v + [K_a K_b/R_a]$	0.1438	Nm sec/rad
K_a/R_a	10.0	Nm/V

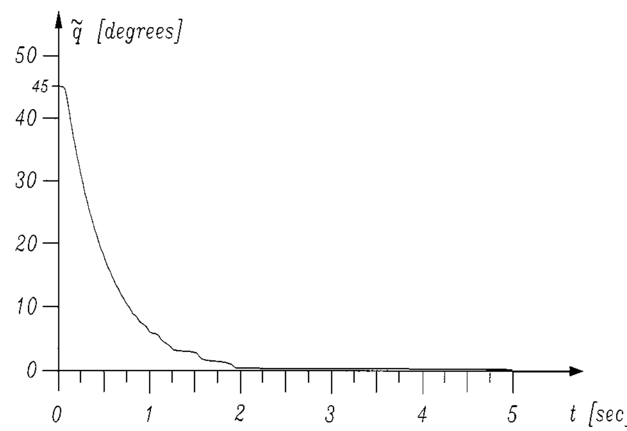

Fig. 4. Position error.

The experiment was carried out under the initial conditions: $q(0) = 0$ and $\dot{q}(0) = 0$. The desired shaft position q_d was 45°, and the controller parameters were

$$\begin{aligned} k &= 2.0 && \text{[1/s]} \\ k_i &= 1.0 && \text{[V/rad]} \\ k_v &= 0.001 && \text{[V s/rad], and} \\ \overline{J} &= 0.001 && \text{[V kg m}^2\text{/[Nm}\cdot\text{ rad]].} \end{aligned}$$

It can be easily checked by straightforward substitution that $\overline{J}$ satisfies condition (23), thus the closed-loop system stability is guaranteed.

The time evolution of the position error $\tilde{q}$ obtained from the experiment is shown in Fig. 4. The position error response presents a fast transient toward a neighborhood of zero. Then, it continues to decrease slowly approaching zero due to the integral action. This situation is a typical behavior of exponentially stable linear systems. A faster response can be achieved by increasing the gain k_v; but it demands higher torques beyond the limit prescribed by the motor manufacturer.

V. Concluding Remarks

The vast majority of regulators in the industry are linear PID controllers. There are many reasons for this selection, including their long history of proven operation, which is well understood by many operational, technical, and maintenance individuals.

The application of PID controllers to regulation of dc motors is widely spread in automation of mechanical processes. Depending on the feedback signals available for measurement, several alternatives for practical implementation structures of PID controllers can be considered. This paper has discussed two basic structures paying attention to stability issues.

The conclusion of this study—which involved automatic control concepts such as transfer function, characteristic polynomial, stability, and Routh–Hurwitz criterion—is that implementation of PID control as a hierarchical control structure invoking a velocity inner loop needs less stringent and easy to check conditions for closed-loop system stability.

As a practical matter, experiments of the PID control of a direct-drive motor were conducted and the results presented in the paper.

APPENDIX

Consider the characteristic polynomial of a third-order system expressed in the Laplace variable s given by

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0. \tag{24}$$

Following the Routh–Hurwitz criterion [2], [7], the array of (24) is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & \end{array}$$

where

$$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} \quad \text{and} \quad c_1 = \frac{b_1 a_0}{b_1} = a_0.$$

For the third-order system to be stable, it is necessary and sufficient [2], [7] that the coefficients a_0, a_1, a_2, and a_3 be positive and $a_2 a_1 - a_0 a_3 > 0$.

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