A Non-Trial-and-Error Method for Lag-Lead Compensator Design

Kai Shing Yeung, *Member, IEEE*, King W. Wong, and Kan-Lin Chen

***Abstract*—A universal design chart for lag-lead compensators is given. This chart allows compensators to be designed without trial-and-error while meeting four specifications: steady-state error, phase margin, gain margin, and gain (or phase) crossover frequency.**

***Index Terms*— Compensator design, control systems, lag-lead compensator.**

I. Introduction

FREQUENCY-domain design in general, or Bode design in particular, is a widely used and fundamental tool for control system design. However, because of the trial-and-error nature of this technique, designing often times will be inconvenient. In 1976, Wakeland [1] developed an analytical technique and a graph pickoff approach for continuous phase-lead compensator. The following year, Mitchell [2] extended Wakeland's work to continuous phase-lag compensators. Wakeland and Mitchell's work was then pursued by Chaid to generalize to discrete and second-order compensators. In 1986, he presented tables of Bode design formulas for the most common continuous and discrete compensators [3], [4]. Recently, Yeung *et al.* developed a series of Bode design charts to allow non-trial-and-error designs of continuous and discrete-time compensators [5].

In this paper, a non-trial-and-error design technique for lag-lead compensator is developed based on the idea presented in [5]. Simultaneous specification of steady-state error, phase margin, gain margin, and gain (or phase) crossover frequency can be achieved using this technique.

II. The Lag-Lead Compensator

Consider a lag-lead compensator of the form

$$G_c(s) = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\gamma}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \tag{1}$$

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K. S. Yeung is with the Department of Electrical Engineering, The University of Texas at Arlington, Arlington, TX 76019 USA.

K. W. Wong was with the Department of Electrical Engineering, The University of Texas at Arlington, Arlington, TX 76019 USA. He is now with GSM Wireless Division, Northern Telecom, Richardson, TX 75082-4399 USA.

K.-L. Chen was with the Department of Electrical Engineering, The University of Texas at Arlington, Arlington, TX 76019 USA. He is now with the Department of Electronics, Fortune Junior College of Technology and Commerce, Kaohsiung, Taiwan, R.O.C.

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where $\gamma = \beta$ is assumed. Equation (1) can be rewritten as

$$\begin{aligned} G_c(s) &= K_c \frac{s^2 + \left(\frac{1}{T_1}+\frac{1}{T_2}\right)s + \frac{1}{T_1T_2}}{s^2+\left(\frac{\beta}{T_1}+\frac{1}{\beta T_2}\right)s+\frac{1}{T_1T_2}} \\ &= K_c \frac{T_1T_2s^2+(T_1+T_2)s+1}{T_1T_2s^2+\frac{\beta^2T_2+T_1}{\beta}s+1}. \end{aligned} \tag{2}$$

Substituting $s = j\omega$ into (2) yields

$$\begin{aligned} G_c(j\omega) &= K_c \frac{1-T_1T_2\omega^2+j(T_1+T_2)\omega}{1-T_1T_2\omega^2+j\frac{\beta^2T_2+T_1}{\beta}\omega} \\ &= K_c \cdot \overline{G}_c(j\omega). \end{aligned} \tag{3}$$

By introducing the parameters

$$T=\sqrt{T_1T_2} \qquad \zeta = \frac{T_1+T_2}{2\sqrt{T_1T_2}} \qquad \tau = \frac{\beta^2T_2+T_1}{\beta(T_1+T_2)} \tag{4}$$

(3) can be rewritten as

$$\overline{G}_c(j\omega) = \frac{1-T^2\omega^2+j2\zeta T\omega}{1-T^2\omega^2+j2\tau\zeta T\omega} = \frac{1+j2\frac{\zeta T\omega}{1-T^2\omega^2}}{1+j2\tau\frac{\zeta T\omega}{1-T^2\omega^2}}.$$

With the definition of

$$\Omega = \frac{\zeta T\omega}{1-T^2\omega^2} \tag{5}$$

$\overline{G}_c(j\omega)$ becomes

$$\overline{G}_c(\tau,\Omega) = \frac{1+j2\Omega}{1+j2\tau\Omega}. \tag{6}$$

$\overline{G}_c(\tau,\Omega)$ can be plotted in terms of magnitude (in decibels) versus phase as a family of curveswith parameters τ and Ω (Fig. 1).

III. Principle of the Design Method

Suppose we propose to design a lag-lead compensator $G_c(j\omega)$ for a plant $G_p(j\omega)$ such that 1) phase margin PM, 2) gain margin GM, and 3) the gain crossover frequency ω_1 are specified. (Although gain and phase margins may not give an accurate prediction of the actual time-domain behavior, they are convenient intermediate frequency-domain specifications.)

Step 1: K_c is assumed to have been determined from the steady-state accuracy specification as usual.

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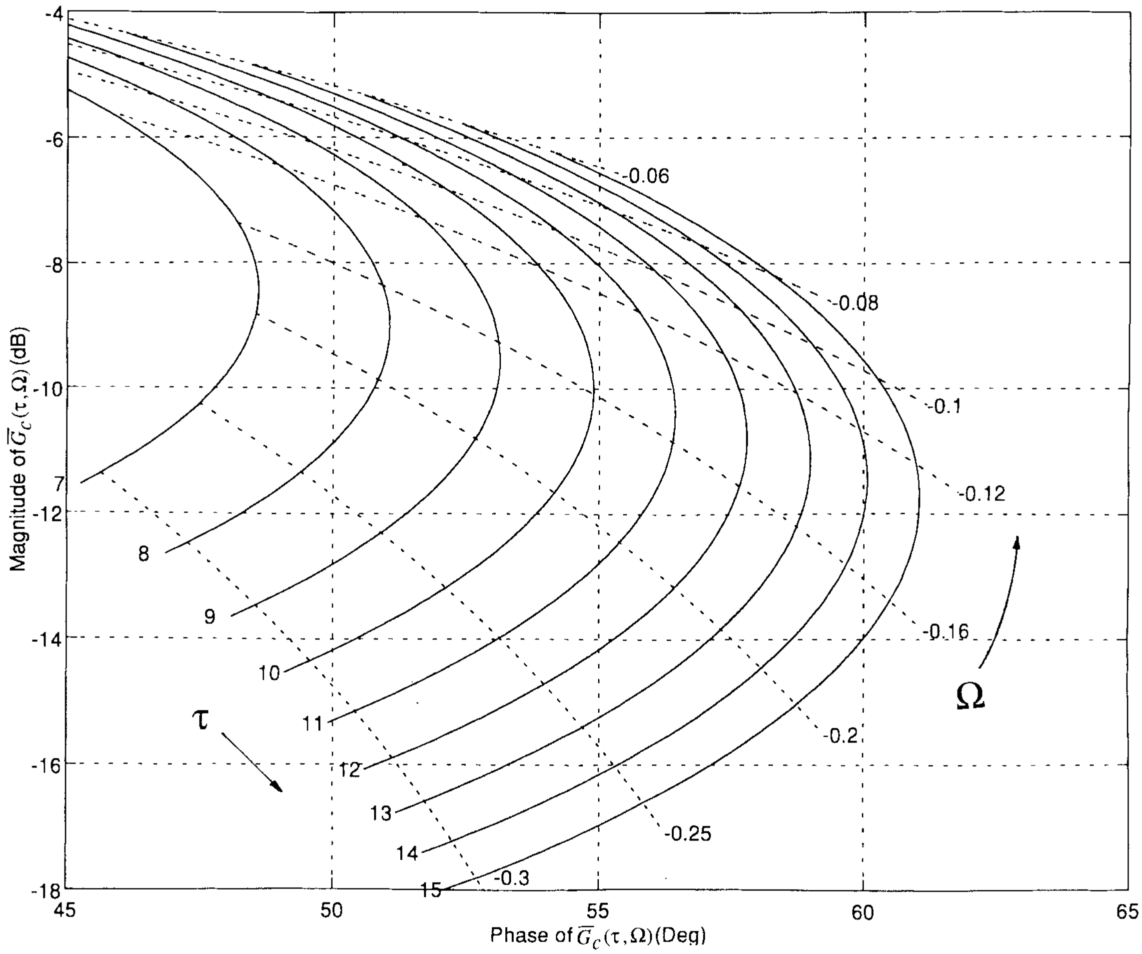

Fig. 1. Lag-lead controller design curves.

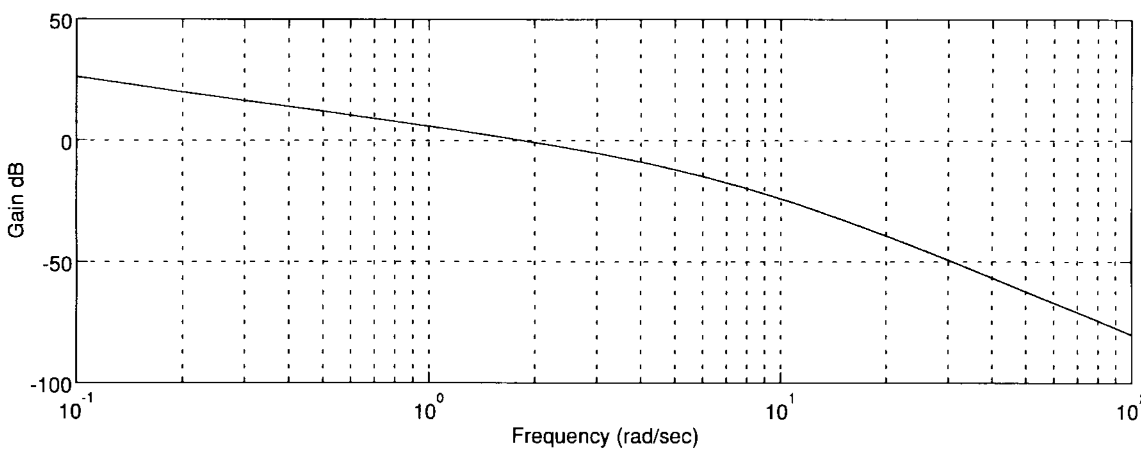

Fig. 2. Bode plot of uncompensated system.

Step 2: If ω_1 is the gain crossover frequency, then the following equations must hold:

$$|G_c(j\omega_1) \cdot G_p(j\omega_1)|_{\rm dB} = 0 \quad (7a)$$
$$\angle(G_c(j\omega_1) \cdot G_p(j\omega_1)) = {\rm PM} - 180^\circ \quad (7b)$$

where $|\cdot|_{\rm dB} = 20 \cdot \log_{10}|\cdot|$.

Substituting (3) and (6) into (7) gives

$$|\overline{G}_c(\tau,\Omega)|_{\rm dB} = -|K_c G_p(j\omega_1)|_{\rm dB} \quad (8a)$$
$$\angle\overline{G}_c(\tau,\Omega) = -\angle(K_c G_p(j\omega_1)) + {\rm PM} - 180^\circ. \quad (8b)$$

The left-hand side of (8) can be plotted once for all time as a lag-lead design chart (Fig. 1). The right-hand side of (8) can be plotted on top of the lag-lead design chart. The

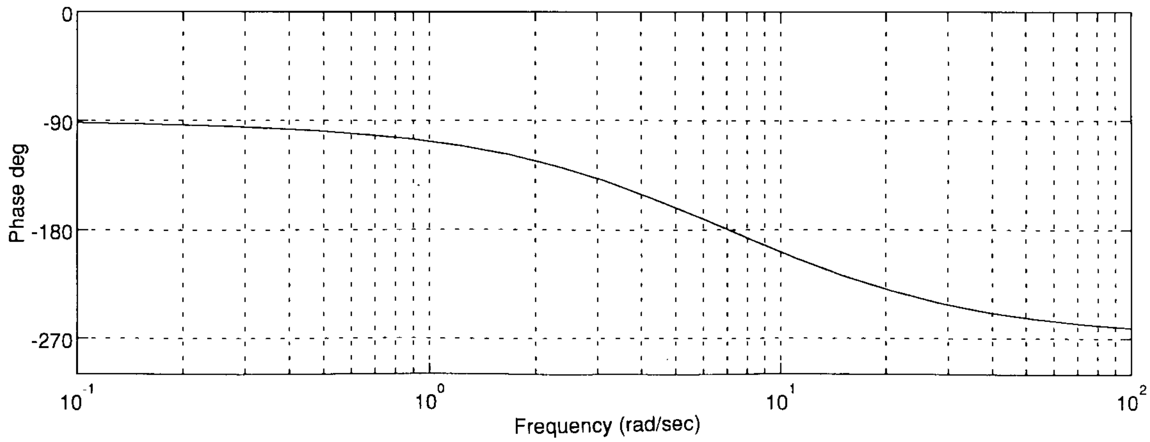

Fig. 2. *(Continued.)* Bode plot of uncompensated system.

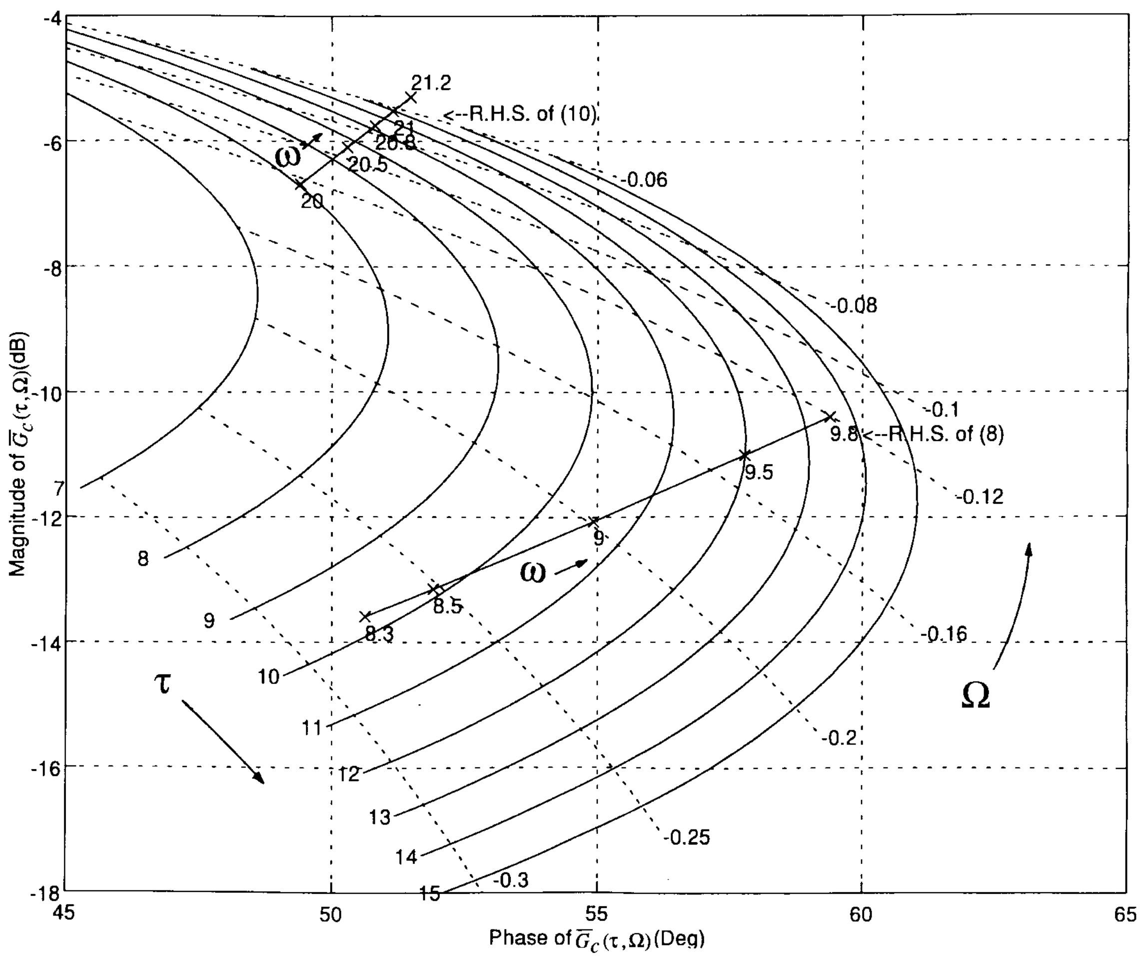

Fig. 3. Illustrating lag-lead controller design.

intersection point with $\omega = \omega$ yields the desired parameter values of $\tau = \tau_1, \Omega = \Omega$.

Step 3: Let ω_2 be the phase crossover frequency. Then the following equations must hold:

$$|G_c(j\omega_2) \cdot G_p(j\omega_2)|_{\text{dB}} = -\text{GM} \tag{9a}$$

$$\angle(G_c(j\omega_2) \cdot G_p(j\omega_2)) = -180^\circ \tag{9b}$$

which can be written from (3) and (6) as

$$|\overline{G}_c(\tau, \Omega)|_{\text{dB}} = -|K_c G_p(j\omega_2)|_{\text{dB}} - \text{GM} \tag{10a}$$

$$\angle\overline{G}_c(\tau, \Omega) = -\angle(K_c G_p(j\omega_2)) - 180^\circ. \tag{10b}$$

Then plotting the right-hand side of (10) on the lag-lead design chart and looking for the intersection point with $\tau = \tau_1$ yields $\omega = \omega_2$ and $\Omega = \Omega_2$.

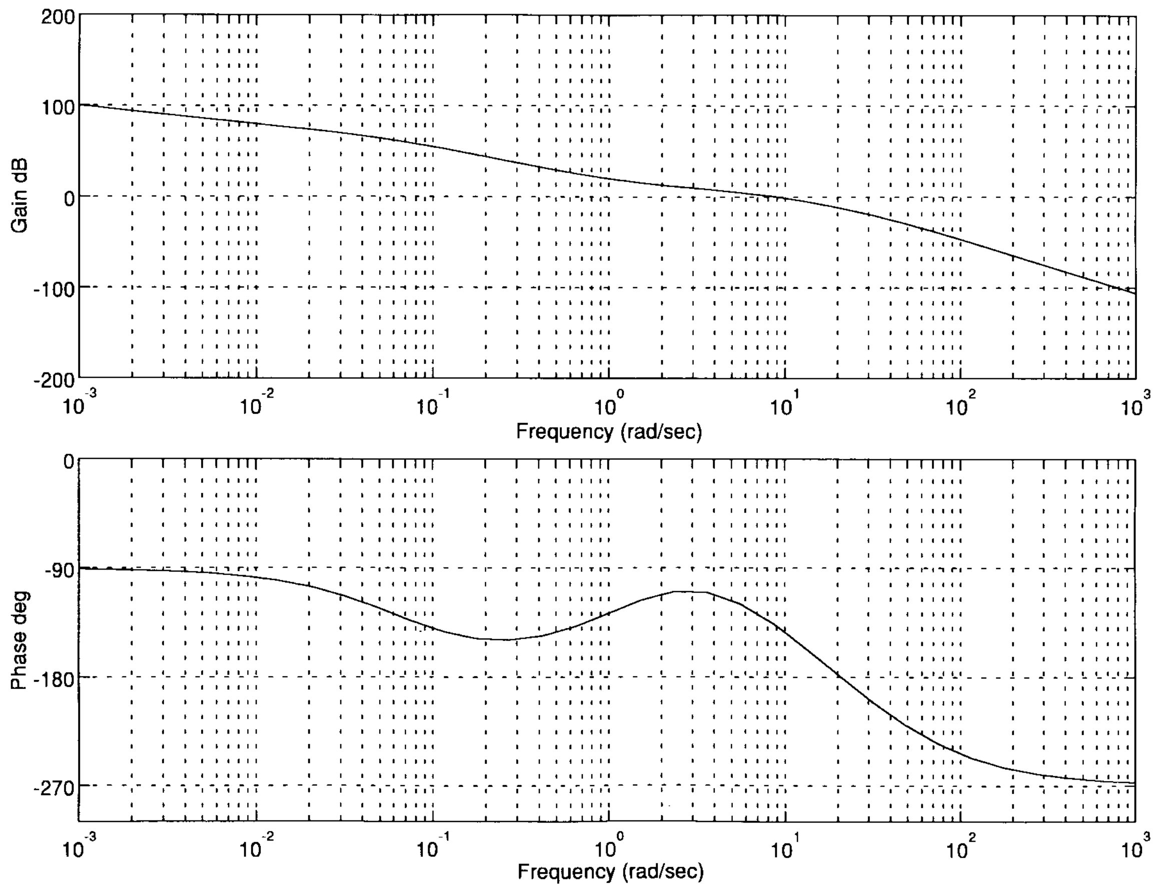

Fig. 4. Bode plot of compensated system.

Step 4: With the five known parameters $\tau_1, \omega_1, \omega_2, \Omega_1$, and Ω_2 obtained from Steps 1–3, the lag-lead compensator parameters in (1) can be determined by (11), shown at the bottom of this page.

Formula (11) is obtained by solving for T_1, T_2, and β from (4) and from the two equations

$$\Omega_1 = \frac{\zeta T \omega_1}{1 - T^2\omega_1^2} \qquad \Omega_2 = \frac{\zeta T \omega_2}{1 - T^2\omega_2^2}$$

which is (5) applied at frequencies $\omega = \omega_1$ and $\omega = \omega_2$, respectively.

The entire procedure will be demonstrated using the following illustrative example.

IV. Illustrative Example

In this example we design a lag-lead compensator for a control system whose open-loop transfer function is

$$G_p(s) = \frac{100}{s(s+5)(s+10)}.$$

It is desired that the static velocity error constant K_v be 100 s^{-1}, the phase margin be 42° at frequency 9 rad/s, and the gain margin be 12 dB.

The Bode plot of the uncompensated system $G_p(s)$ is shown in Fig. 2.

Step 1: Determination of K_c

$$\begin{aligned} K_\nu &= \lim_{s\to 0} sG_c(s)G_p(s) \\ &= \lim_{s\to 0} sG_c(s)\frac{100}{s(s+5)(s+10)} = 2K_c = 100 \\ K_c &= 50. \end{aligned}$$

Step 2: Plot the right-hand side of (8) (solid line) on the lag-lead design chart as shown in Fig. 3. The intersection point corresponding to $\omega = \omega_1 = 9$ rad/s gives

$$\begin{aligned} \tau &= \tau_1 = 10.5 \\ \Omega &= \Omega_1 = -0.2. \end{aligned}$$

$$\begin{aligned} T_1 &= \frac{\Omega_1\Omega_2(\omega_2^2-\omega_1^2) - \sqrt{(\Omega_1\Omega_2(\omega_2^2-\omega_1^2))^2 - \omega_1\omega_2(\omega_1\Omega_2-\omega_2\Omega_1)(\omega_2\Omega_2-\omega_1\Omega_1)}}{\omega_1\omega_2(\omega_2\Omega_2-\omega_1\Omega_1)} \\ T_2 &= \frac{\Omega_1\Omega_2(\omega_2^2-\omega_1^2) + \sqrt{(\Omega_1\Omega_2(\omega_2^2-\omega_1^2))^2 - \omega_1\omega_2(\omega_1\Omega_2-\omega_2\Omega_1)(\omega_2\Omega_2-\omega_1\Omega_1)}}{\omega_1\omega_2(\omega_2\Omega_2-\omega_1\Omega_1)} \\ \beta &= \frac{\tau(T_1+T_2) + \sqrt{\tau^2(T_1+T_2)^2 - 4T_1T_2}}{2T_2} \end{aligned} \tag{11}$$

(Note that changing the phase margin specification amounts to a horizontal shift of the solid line.)

Step 3: Plot the right-hand side of (10) (solid line) on the lag-lead design chart.

The intersection point corresponding to $\tau = \tau_1 = 10.5$ gives

$$\omega = \omega_2 = 20.65$$
$$\Omega = \Omega_2 = -0.085.$$

(Note that altering the gain margin will give rise to a vertical shifting of the solid line.)

Step 4: Substituting the value of $\tau_1, \omega_1, \omega_2, \Omega_1$, and Ω_2 into (11) yields

$$T_1 = 0.4$$
$$T_2 = 1.013$$
$$\beta = 14.167.$$

The lag-lead compensated system is

$$G(s) = G_c(s)G_p(s) = \left[\frac{50 \cdot (s+2.5)(s+0.987)}{(s+36.54)(s+0.0675)}\right] \cdot \left[\frac{100}{s(s+5)(s+10)}\right].$$

The Bode plot of the compensated system $G(s)$ in Fig. 4 (on this and the following page) shows a phase margin PM $= 42°$ at a gain crossover frequency $\omega_1 = 9$ rad/s, and a gain margin GM $= 12$ dB at a phase crossover frequency $\omega_2 = 20.65$ rad/s.

V. Conclusion

A non-trial-and-error method for lag-lead compensator design is presented. It is based on the idea of normalizing compensator parameters so that a design chart can be generated once for all time. A design example is carried through to illustrate the use of this design chart. Simultaneous fulfillment of the specifications of steady-state error, phase margin, gain margin, and gain crossover frequency (or instead, phase crossover frequency) can be achieved without trial-and-error.

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Kai Shing Yeung (M'81) was born in Foshan, Guangdong, China, on November 2, 1948. He received the B.Sc. degree from the University of Hong Kong in 1970, the Dipl.-Ing. and Dr.-Ing. degrees (both *summa cum laude*) from the University of Karlsruhe, Karlsruhe, Germany, in 1974 and 1977, respectively, all in electrical engineering.

From 1970 to 1971, he served as an Assistant Project Engineer at the Hong Kong Electric Company and was involved with the design of a 3/11-kV zone station and the design of 132-kV switching stations. From 1974 to 1977, he was a Research Associate at the Institute for Automatic Control, University of Karlsruhe, where he was engaged in the study of nonlinear vibrations. From 1977 to 1983, he served as a Lecturer in Electronics at the Chinese University of Hong Kong. During his leave of absence in 1983, he was a Visiting Associate Professor in the Department of Mechanical Engineering and Mathematics, Drexel University, Philadelphia, PA. In Fall of 1983, he assumed the position at the Electrical Engineering Department, The University of Texas at Arlington, where he is currently a Professor. From 1991 to 1994, he was involved in beam transfer control at the Superconducting Super Collider Laboratory. His research interests include feedback control systems and signal processing.

King W. Wong received the B.S.E.E. degree (with honors) from The University of Texas at Arlington in 1995.

He is currently employed at Northern Telecom, GSM Wireless Division, as a Member of Scientific Staff and is also a graduate student at the University of Texas at Dallas. His interest is in digital signal processing systems.

Kan-Lin Chen was born in Taiwan, R.O.C., in 1956. He graduated from the Taipei Institute of Technology, Taipei, R.O.C., in 1979. He received the B.S. and M.S. degrees from Utah State University, Logan, in 1987 and 1989, respectively, and the Ph.D. degree from the University of Texas at Arlington in 1995, all in electrical engineering.

From 1981 to 1983 he was an engineer in Taiwan Tandy electronic company. Since 1996, he has been with the Department of Electronics, Fortune Junior College of Technology and Commerce, Kaohsiung, Taiwan, R.O.C., where he is currently an Associate Professor. His research interests are in control systems and signal processing.