Root Loci Design Using Dickson's Technique

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Abstract—Traditional techniques for determining the root loci were developed by Evans. Given the characteristic equation $\{1 + G(s)H(s) = 0\}$, Evans root locus method requires the magnitude of $G(s)H(s)$ to be equal to minus one and is well established for determining the pole locations as the system gains are changed. In this paper, nontraditional techniques for developing the root loci of control systems are introduced. Illustrations are developed by using a method for plotting the root loci that involves setting the real and the imaginary portions of the characteristic equation equal to zero. Setting the real and imaginary parts of the characteristic equation equal to zero arises from work in the theory of equations by Dickson. Application of this technique allows adjustment of the gain, to establish the system requirements, and introduces collateral equations subject to unique geometric interpretation.

Index Terms—Characteristic equation, control systems, root locus techniques, theory of equations.

I. INTRODUCTION

A N IMPORTANT topic of linear control systems is the investigation of the trajectories of the roots of the characteristic equation. Given such an equation, with K as a variable, an engineer wanting to see the locus might pick a value for K, then find and plot the roots. This process can be repeated, for different values of K, until the desired locus is available for viewing. In general, this was not a viable option when the root locus principles were first introduced by Evans [1], [2]. His work enabled the engineer to bypass such a cumbersome procedure and quickly obtain a sketch of the root locus. The Evans' principles are still useful for that purpose.

Evans' root loci rules are reviewed in order to demonstrate construction of the root loci for the characteristic equation of a typical feedback control system as shown in Fig. 1. The characteristic equation is obtained by setting $1 + G(s)H(s) = 0$. The rules for actually constructing the loci are summarized by Kuo [4] and presented in Table I.

Engineers readily construct the system's root loci by using the theorems developed by Evans; however, the method does not provide an analytical technique for determining the required gains. By following the techniques of Dickson [3], equations are developed that can lead to a different understanding of how the system behaves. These equations can be used to set the value of gain to meet the requirements for the system and also provide algebraic expressions for the loci.

The first goal of this paper is to review the Evans root locus method. Then the second goal is to introduce the Dickson technique and show how it can be utilized as a supplemental tool for constructing the desired loci. Also applications where the gain
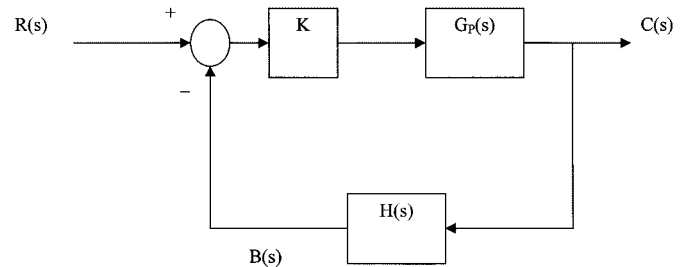
Fig. 1. Controlled system.

can be precisely determined will be illustrated and the geometric nature of the collateral equations will be discussed.

II. EVAN'S METHOD—THE TRADITIONAL APPROACH

The nine theorems in Table I summarize the traditional approach presented by Evans. By way of demonstration, a problem discussed in Kuo [4] is presented.

Example 1: Considering $G(s) = K/s(s+1)$ and $H(s) = 1$, the characteristic equation $(1 + G(s)H(s) = 0)$ becomes

$$s(s + 1) + K = 0. \tag{1}$$

From Theorem 1, in Table I, when $K = 0$, the two roots of the equation are $s = 0$ and $s = -1$. These two points are the poles shown in Fig. 2.

From the given information

$$1 + G(s)H(s) = 1 + (K/(s(s + 1))) = 0, \tag{2}$$

and

$$G(s)H(s) = K/(s(s + 1)). \tag{3}$$

Given (1) it is apparent from Theorem 2, in Table I, that when K equals $-\infty$ or $+\infty$ then s must equal ∞ or $-\infty$, respectively. Thus, the system poles are located at $-\infty$ and $+\infty$.

From Theorem 3, the order of the polynomial in (1) is two; therefore the number of branches of the root loci (RL) is also two.

The RL plot is determined as the system gain K is varied from 0 to ∞. The complimentary root loci (CRL) plot is determined as the gain is varied from 0 to $-\infty$. The combination of the RL and the CRL make up the total or complete root loci.

From Theorem 5, the angles of the asymptotes for the RL portion are at 90° and 270°. The CRL angles are calculated to be 0° and 180°.

From Theorem 6, the intersection of the asymptotes is determined to be $\sigma_1 = -1/2$.

Theorem 7 indicates that the real axis between the two poles is part of the RL plot and the remainder of the real axis belongs to the CRL plot.

TABLE I
PROPERTIES OF THE COMPLETE ROOT LOCI [4]

Theorem 1. The K = 0 points on the complete root locus are at the poles of G(s)H(s).

Theorem 2. The K = +/- ∞ points on the complete root loci are at the zeros of G(s)H(s).

Theorem 3. The number of branches of the root loci is equal to the order of the polynomial.

Theorem 4. The complete root loci are symmetrical with respect to the real axis of the s-plane. In general, the loci are symmetrical with respect to the axes of symmetry of the pole-zero configuration of G(s)H(s).

Theorem 5. For large values of s, the RL (K ≥ 0) are asymptotic with angles given by

$$\theta_K = \frac{(2K + 1)\pi}{|n - m|} \qquad n \neq m$$

where K = 0, 1, 2, . . . , $|n - m|$ - 1; n and m are the number of finite poles and zeros of G(s)H(s), respectively.
For the CRL (K ≤ 0), the angles of the asymptotes are

$$\theta_K = \frac{2K\pi}{|n - m|} \qquad n \neq m$$

where K = 0, 1, 2, . . . $|n - m|$ - 1.

Theorem 6. (a). The intersect of the $2|n - m|$ asymptotes of the complex root loci lies on the real axis of the s-plane.
   (b). The intersect of the asymptotes is given by

$$\sigma_1 = \frac{\Sigma \text{ finite poles of G(s)H(s)} - \Sigma \text{ finite zeros of G(s)H(s)}}{n - m}$$

where n is the number of finite poles of G(s)H(s), and m is the number of finite zeros of G(s)H(s).

Theorem 7. The entire real axis of the s-plane is occupied by the complete root loci; i.e., either the RL or CRL.

A. RL: On a given section of the real axis, RL are found in the section only if the total number of poles and zeros of G(s)H(s) to the right of the section is odd.

B. CRL: On a give section of the real axis, CRL are found in the section only if the total number of real poles and zeros of G(s)H(s) to the right of the section is even.

Complex poles and zeros of G(s)H(s) do not affect the distribution of the root loci on the real axis.

Theorem 8. The breakaway points on the complete root loci of 1 + KG(s)H(s) = 0 must satisfy

$$\frac{dG(s)H(s)}{ds} = 0$$

Theorem 9. The intersection of ω and K at the crossing points of the root loci (-∞ < K < ∞) on the imaginary axis of the s-plane may be determined by use of the Routh-Hurwitz criterion.

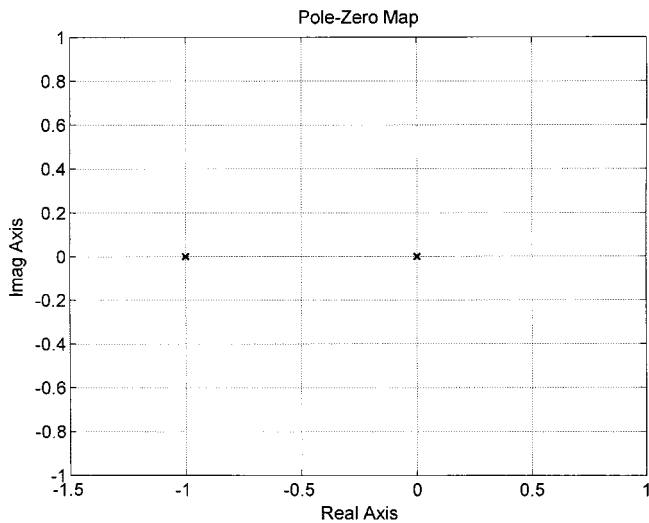Fig. 2.   Pole zero plot of Example 1.

Fig. 3.   Root loci plot of Example 1.

To find the breakaway points on the complete root loci, Theorem 8 is applied. Taking the derivative of $G(s)H(s)$ with respect to s, the following is observed:

$$\frac{dG(s)H(s)}{ds} = \frac{-2s - 1}{s^4 + 2s^3 + s^2} = 0$$

or

$$-2s - 1 = 0. \tag{4}$$

The root of (4) is at $s = -0.5$ which indicates that the breakaway point and the intersection of the asymptotes are identical for this system. To determine the system gain at the breakaway point, $s = -0.5$ is inserted into Equation (1) which yields $K = 0.25$.

Applying the Routh–Hurwitz criterion as suggested in Theorem 9, it is found that $K > 0$ is required to maintain stability. A negative system gain will cause the roots to cross into the right-half plane, thus causing instability.

With the results from the nine Theorems, the root loci for the system can be constructed. Plots of the loci are displayed in Fig. 3 for RL and Fig. 4 for CRL.

III. THE DICKSON METHOD

Using Evan's method, the characteristic equation, $1 + G(s)H(s) = 0$, is set equal to zero using the following conditions:

$$|G(s)H(s)| = 1 \quad \text{and} \quad \angle G(s)H(s) = 180°.$$

The system shown in Fig. 1 is to be considered. If the system transfer function, $C(s)/R(s) = G(s)H(s)/(1 + G(s)H(s))$, is simplified to a ratio of two polynomials, i.e.,

$$C(s)/R(s) = b(s)/a(s),$$

then the polynomial closed-loop characteristic equation can be defined as
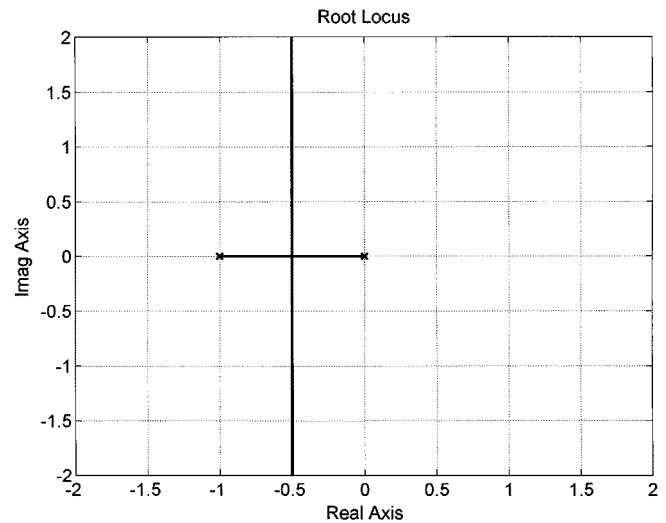
$$a(s) = F(s) + 1 = 0.$$

For this case, the root locus conditions are established by requiring

$$\text{Re}\,[F(s)] = -1 \quad \text{and} \quad \text{Im}\,[F(s)] = 0$$

which, for $s = \sigma + j\omega$, can be written as follows:

$$\text{Re}\,[F(\sigma + j\omega)] = -1 \tag{5}$$

and

$$\text{Im}\,[F(\sigma + j\omega)] = 0. \tag{6}$$

This is the contribution of Dickson [3]. Equation (5) is denoted the real condition and (6) the imaginary condition.

Referring to Example 1

$$C(s)/R(s) = K/(s^2 + s + K) = K/(F(s) + 1) \tag{7}$$

or

$$F(s) = s^2 + s + K - 1, \tag{8}$$

and

$$F(\sigma + j\omega) = (\sigma + j\omega)^2 + (\sigma + j\omega) + K - 1$$
$$= \sigma^2 - \omega^2 + \sigma + K - 1 + j\omega(1 + 2\sigma). \tag{9}$$

The Imaginary Condition applied to (9) results in the following:

$$\omega(1 + 2\sigma) = 0. \tag{10}$$

This produces $\omega = 0$ or $\sigma = -1/2$. Plotting this outcome reveals the complete root loci of the system. Notice that ω equal to zero indicates that part of the RL and part of the CRL locus encompasses the σ axis (this is the situation for all systems). The other result from (10), $(\sigma = -1/2)$, indicates that the loci occupy the line one-half a unit to the left of the ω axis. The results can be seen in Fig. 3.

Examining Fig. 3, the RL, from the Evans approach, it is known that the poles are located at (0,0) and $(-1,0)$ when the system gain $K = 0$. The loci are bound between the two poles and the value $\sigma = -1/2$ as K varies from 0 to ∞. Fig. 4, indicates that the CRL occupies the remainder of the real axis.

Applying the real condition to (9)

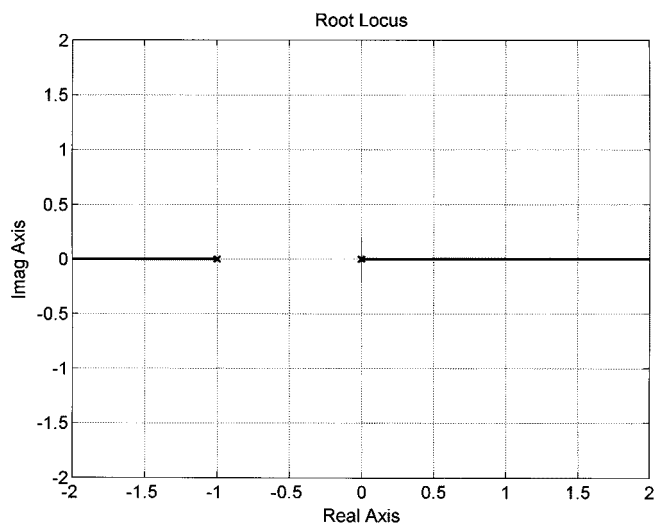$$K = \omega^2 - \sigma^2 - \sigma. \tag{11}$$

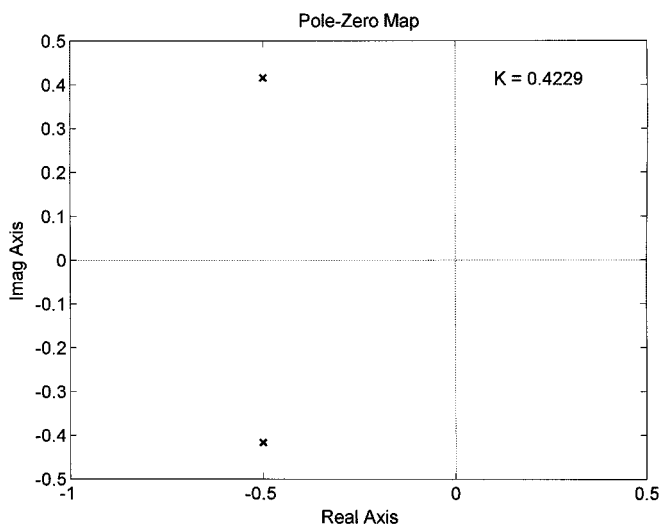Fig. 4.   Complimentary root locus plot of Example 1.

Fig. 5.   Pole zero plot of Example 1 with system gain $K = 0.4229$.

In Equation (11), it will be noticed that the system gain can be determined from the values of σ and ω. Thus the necessary gain, K, can be calculated, to place the roots where desired on the loci. If a system pole is to be located at the point $(-0.5, 0.4158)$, inserting these values into (11) will produce a system gain of 0.4229. The pole zero plot of the system with a gain of 0.4229 is illustrated in Fig. 5. One pole is located at $(-0.5, j0.4158)$, while the other is at the point $(-0.5, -j0.4158)$. So, the system poles can be placed on the root locus by determining the proper system gain from (11).

Equations (10) and (11) are derived using Dickson's technique to obtain an imaginary and a real condition. Evan's approach evolves from setting $G(s)H(s) = -1$. Both methods (Evans and Dickson) determine the proper root locus. The Dickson approach also exploits a calculation for the gain (K) to place the system poles at the desired operating point.

The implementation of the root locus method could involve picking K, then finding and plotting the roots of the characteristic equation for the system as K varies. Equation (10) indicates that if this is completed for all K, then the plot will be on the

real axis for $\omega = 0$, or the plot will be on the line $\sigma = -1/2$ for values of ω other than zero. The analytical results illustrated by (10) are duplicated by developing the root locus plot. Yet, when using both (10) and (11) the designer has more options to consider than are available when using the root locus rules of Evans.

Example 2: Consider another system discussed by Kuo [4] and Ogata [5], where

$$G(s) = K/(s(s+1)(s+2)) \quad \text{and} \quad H(s) = 1. \quad (12)$$

The closed-loop transfer function can be written as

$$\begin{aligned} C(s)/R(s) &= K/(s^3 + 3s^2 + 2s + K) \\ &= K/(s^3 + 3s^2 + 2s + K - 1 + 1) \\ &= K/(F(s) + 1) \end{aligned}$$

where

$$F(s) = (s^3 + 3s^2 + 2s + K - 1). \quad (13)$$

Inserting $s = \sigma + j\omega$ yields

$$F(\sigma + j\omega) = (\sigma + j\omega)^3 + 3(\sigma + j\omega)^2 + 2(\sigma + j\omega) + K - 1. \quad (14)$$

Expanding terms of (14) into real and imaginary parts results in

$$\begin{aligned} F(\sigma + j\omega) = {}&\sigma^3 - 3\sigma\omega^2 + 3\sigma^2 - 3\omega^2 + 2\sigma + K - 1 \\ &+ 3j\omega\sigma^2 - j\omega^3 + 6j\sigma\omega + 2j\omega. \end{aligned} \quad (15)$$

Applying the imaginary condition, the following is obtained from (15):

$$\omega(3\sigma^2 - \omega^2 + 6\sigma + 2) = 0. \quad (16)$$

Once again $\omega = 0$ is a solution (the loci will cover the σ axis). And from the factor in parenthesis in (16), the user can obtain

$$3\sigma^2 + 6\sigma + 2 = \omega^2 \quad (17)$$

or

$$3(\sigma + 1)^2 - 1 = \omega^2$$

or

$$(\sigma + 1)^2/(1/\sqrt{3})^2 - \omega^2 = 1. \quad (18)$$

Equation (18) is in the form of a hyperbola centered at $(-1,0)$. This is the root loci equation for the system; the plots can be seen in Figs. 6 and 7. Fig. 6 is the RL plot while Fig. 7 shows the CRL plot.

From the real condition, applied to (15)

$$K = -\sigma^3 + 3\sigma\omega^2 - 3\sigma^2 + 3\omega^2 - 2\sigma. \quad (19)$$

We now have a system equation in terms of gain, σ and ω, and can once again place the poles according to predetermined value for σ and ω. The breakaway points (BPs) can also be determined for the RL and CRL plots. Setting $\omega = 0$ in (18) produces $\sigma_{BP1} = -0.4226$, $\sigma_{BP2} = -1.577$. By inserting these values into (19), the gain of the system at these points is determined. At the point $(-0.4226, 0)$, the system gain is $K_{BP1} = 0.3849$. Since the gain is positive, this point corresponds to the RL plot.
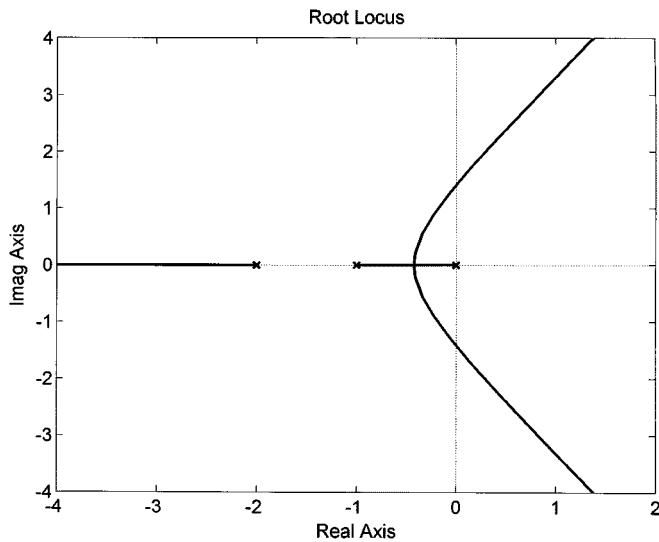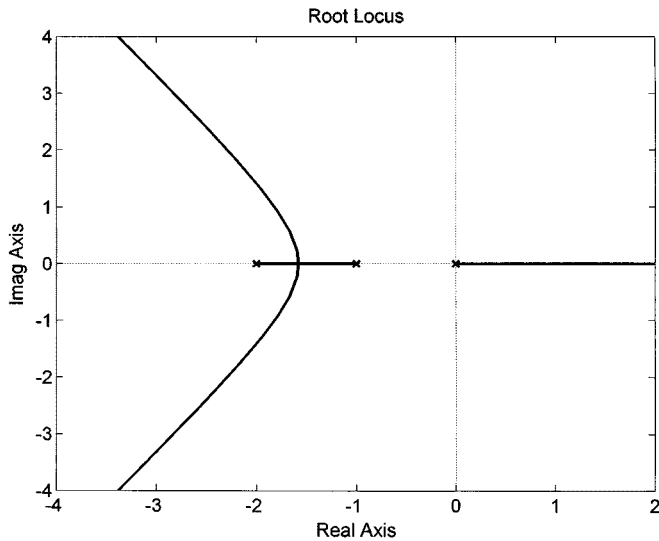
Fig. 6.   Root locus plot of example 2.

Fig. 7.   Complimentary root locus plot of example 2.

At the point $(-1.577, 0)$ on the CRL plot, the system gain is $K_{BP2} = -0.3849$.

The critical values of K can be found from the application of Routh's rule (i.e., $0 < K < 6$ for stability). An examination of the characteristic equation shows that K must be greater than zero, since all the coefficients must be positive; setting $K_C = 0$ and $\sigma_C = 0$ in (19), returns the value for the critical frequency of $\omega_C = 0$. The values, $K_C = 6$ and $\omega_C = 0$, $\sqrt{2}$, are also found from (17) and (19) by setting $\sigma = 0$.

Once again, by developing the system equations using Dickson's technique, the authors are able to determine the plot of the complete root loci. They are also able to determine other important system parameters at points of interest on the plot. This determination can be illustrated by calculating an operating point such that the second order damping ratio (δ) is equal to one-half if $\omega_0^2 = 3\sigma_0^2$. Then from Equation (17) the operating point is located at $\sigma_0 = -1/3$ and $\omega_0 = 0.57735$ while the gain at the operating point, K_0, is calculated from

(19) to be 1.003 70. Other operating points may be implemented using the same technique.

Example 3: For this example $G(s) = (K(s+2))/(s^2+2s+3)$ and $H(s) = 1$. The closed-loop transfer function is

$$C(s)/R(s) = (K(s+2))/(s^2 + (K+2)s + 2K + 3).$$

(20)

Rearranging the terms of (20)

$$C(s)/R(s) = (K(s+2))/(s^2 + (K+2)s + 2K + 2 + 1)$$
$$= (K(s+2))/(F(s) + 1)$$

where $F(s) = s^2 + (K+2)s + 2K + 2$. Inserting $s = \sigma + j\omega$ and reducing yields

$$F(\sigma+j\omega) = \sigma^2 - \omega^2 + (K+2)\sigma + (2K+1) + 2j\omega\sigma + j\omega(K+2).$$

(21)

Applying the imaginary condition to (21)

$$2\omega\sigma + \omega(K+2) = 0$$

(22)

where once again ω is equal to zero or

$$K + 2 + 2\sigma = 0.$$

(23)

Solving for σ in (23) produces

$$\sigma = (-K/2) - 1 = -[(K+2)/2].$$

(24)

Applying the real condition to (21)

$$\sigma^2 - \omega^2 + (K+2)\sigma + 2K + 2 = -1$$

or

$$\sigma^2 + (K+2)\sigma + [(K+2)/2]^2 + 2K + 2$$
$$- [(K+2)/2]^2 - \omega^2 = -1$$

or

$$[\sigma + (K+2)/2]^2 + 2K + 2$$
$$- [(K+2)/2]^2 - \omega^2 = -1$$

or

$$[\sigma + (K+2)/2]^2 + K + 2 + K + 2$$
$$- [(K+2)/2]^2 - \omega^2 = 1.$$

(25)

From before

$$\sigma = -((K+2)/2)$$

or

$$-2\sigma = K + 2.$$

Inserting this into (25) and completing the square results in

$$(\sigma + 2)^2 + \omega^2 = 3.$$

(26)

This is an equation of a circle centered at $\sigma = -2$ and $\omega = 0$ with a radius of $\sqrt{3}$ and is the system equation that describes the RL. The RL plot is shown in Fig. 8, and the CRL plot in Fig. 9.

With (25) and (26), the breakaway points and the gain of the system at these points can be determined. Inserting $\omega = 0$ into (26) produces the breakaway points $\sigma = -0.268$ and -3.73. At the breakaway point $\sigma = -0.268$, K is determined to be -1.464 by examining (24). When $\sigma = -3.73$, K is found to be 5.46.

The critical values of K can be found from the application of Routh's rule (i.e., $2K + 3 > 0$ and $K > -3/2$ for stability).
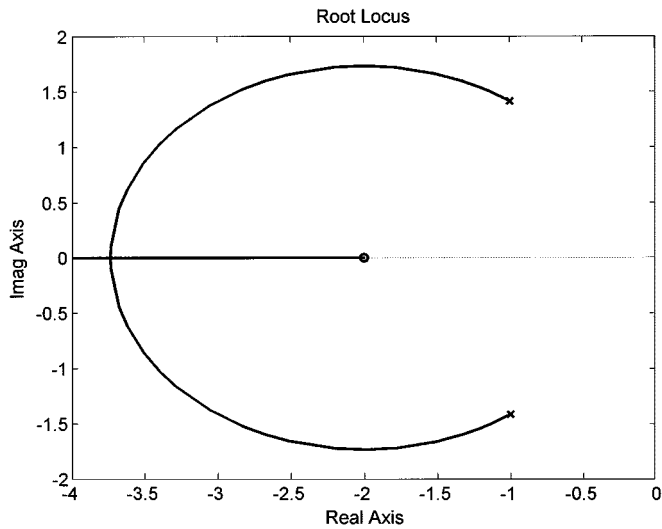
Fig. 8.   Root loci plot of example 3.
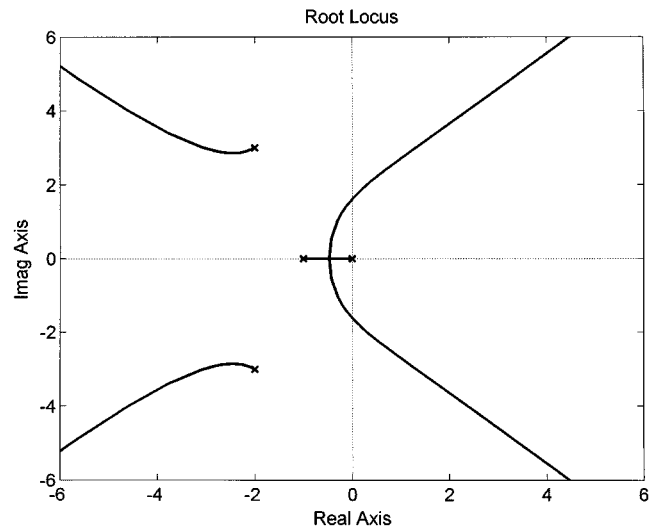

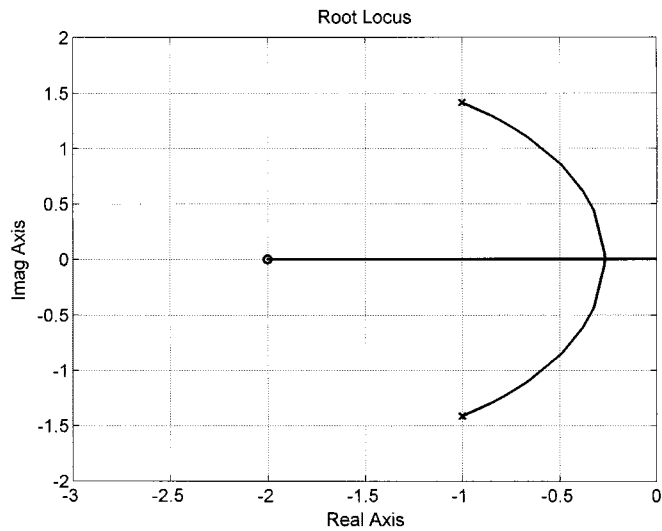Fig. 10.   Root loci plot of example 4.

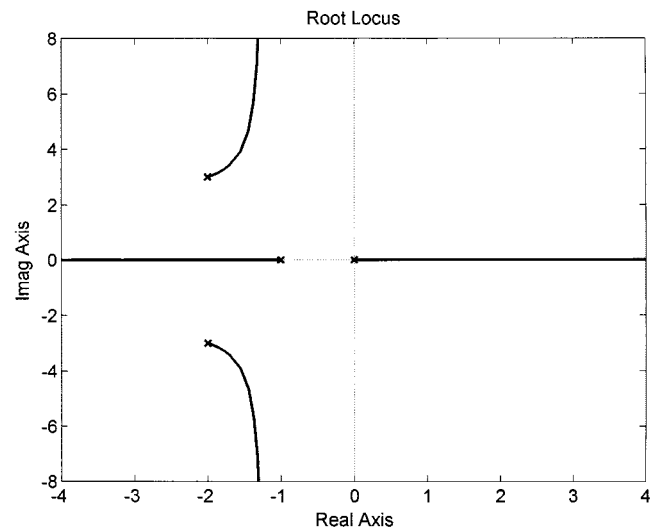Fig. 9.   Complimentary root loci plot of example 3.

Fig. 11.   Complimentary root loci of example 4.

An examination of the characteristic equation shows that these conditions are necessary to keep the coefficients positive. The critical value of K can also be found by evaluating the real condition, knowing that $\sigma = \omega = 0$ at the critical point. If the operating point is to be located with a damping ratio equal to one-half, then this is achieved with a positive feedback gain of $K_o = -1$, $\sigma_o = -1$, and $\omega_o = 0.8660$.

Example 4: A higher order example by Ogata is examined [5] where:

$$G(s) = K/\big(s(s+1)(s^2+4s+13)\big) \quad \text{and} \quad H(s) = 1$$

such that

$$C(s)/R(s) = K/(s^4 + 5s^3 + 17s^3 + 13s + K). \quad (27)$$

For this transfer function, the real part of $F(\sigma+j\omega)$ is calculated to be

$$\mathrm{Re}F(\sigma+j\omega) = \sigma^4 + 5\sigma^3 + 17\sigma^2 + 13\sigma - 6\omega^2\sigma^2$$
$$- 15\omega^2\sigma - 17\omega^2 + \omega^4 + K - 1 \quad (28)$$

And the imaginary part is found as

$$\mathrm{Im}[F(\sigma+j\omega)] = \omega(4\sigma^3 + 15\sigma^2 + 34\sigma + 13 - 4\omega^2\sigma - 5\omega^2). \quad (29)$$

Equations (28) and (29) represent higher order relationships than we have encountered previously; reduction into identifiable geometric curves may not be possible. The root locus plots, RL is shown in Fig. 10 and CRL in Fig. 11, readily show the general shape and the asymptotes for these curves. The real and imaginary conditions applied to the above equations will still provide crucial information needed for analysis of the system and also for establishing the gain, K.

Applying the imaginary condition to (29)

$$\omega(4\sigma^3 + 15\sigma^2 + 34\sigma + 13 - 4\omega^2\sigma - 5\omega^2) = 0 \quad (30)$$

where it is noted that $\omega = 0$ is a solution of the total root locus as expected from the previous examples, and the breakaway point can be found from the enclosed expression, with $\omega = 0$, i.e.,

$$4\sigma^3 + 15\sigma^2 + 34\sigma + 13 = 0.$$

Noting that the roots of the above equation are at $-1.6418 \pm j2.067$ and at -0.4664, the breakaway point is recognized to be at $\sigma_{BP} = -0.4664$. Knowing the value of σ at the breakaway point, the value of the gain, at that point is easily calculated from applying the real condition to (28), and then solving for $K = K_{BP}$, $\sigma = \sigma_{BP}$, and $\omega = 0$.

The real condition requires that

$$\sigma^4 + 5\sigma^3 + 17\sigma^2 + 13\sigma - 6\omega^2\sigma^2 - 15\omega^2\sigma - 17\omega^2 + \omega^4 + K = 0 \tag{31}$$

and with $\sigma = -0.4664$ and $\omega = 0$, $K_{BP} = 2.8252$ from the above equation.

The imaginary condition, (30), was utilized to determine the breakaway point with omega set to zero; it also will provide the critical frequency when sigma is zero, i.e.,

$$5\omega_c^2 = 13 \Rightarrow \omega_c = \pm j1.6125.$$

The real condition, (31), provided the gain at the known breakaway point, σ_{BP}. It also will provide the gain, K_c, at the point of instability when sigma is zero and $\omega = \omega_c$, i.e.,

$$K_c = -\omega_c^4 + 17\omega_c^2 = 37.44.$$

Application of the real and imaginary conditions allowed important points of the root locus plots to be determined. These equations can also help establish the operating point, ω_0 and σ_o, and the associated gain, K_o. By selecting a suitable damping factor, such as $\delta = 1/2$, then $\omega_0/\sigma_o = -\sqrt{3}$, and elimination of ω_0 from (30) results are found in the following relation:

$$-8\sigma_o^3 + 34\sigma_o + 13 = 0$$

with solutions for σ_o as 2.2312, -1.8341, or -0.3971. Since the value for σ_o is expected to be less than the breakaway point of -0.4664, then $\sigma_o = -0.3971$ is chosen as the desired value and ω_0 is calculated to be 0.6878 from the relation $\omega_0/\sigma_o = -\sqrt{3}$. Finally, K_o is calculated to be 8.2179 using the above values in (31).

This example illustrates the application of Dickson's technique to a higher order problem, one where the root locus could not be defined by simple geometric relations. Yet, the fundamental equations established through application of the real and the imaginary conditions are still useful in finding the important factors required in the design and analysis of the system. Exact values were found for the points that breakaway from the real axis, the critical values for oscillation, and the required system gain to achieve the desired operating point. Only algebraic equations were involved and the most difficult mathematical operation performed was finding roots for these equations. The technique is a valuable compliment to classical root locus analysis.

Example 5: The final example involves a fifth-order system as defined by (32).

$$G(s) = K(s^2 + 2s + 4)/s(s + 4)(s + 6)(s^2 + 1.4s + 1)$$

and

$$H(s) = 1. \tag{32}$$

Increasing the number of zeros in $G(s)$ increases the complexity of the problem. Since only algebraic operations are required to complete an analysis using Dickson's technique, higher order systems utilize the same procedures simply using more terms.
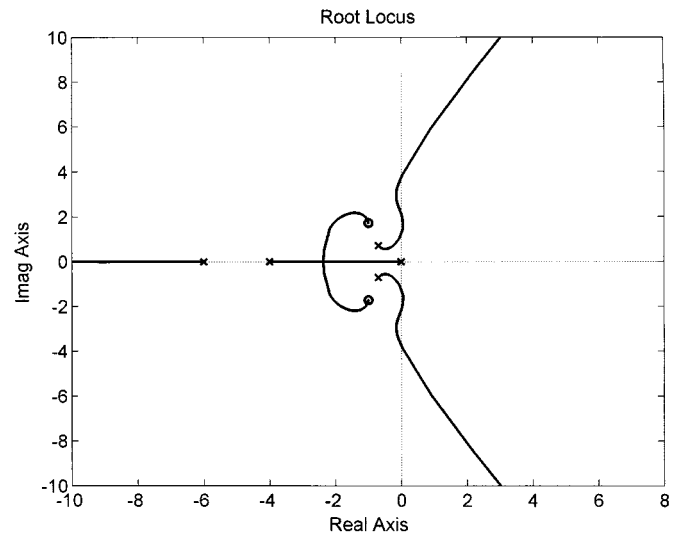

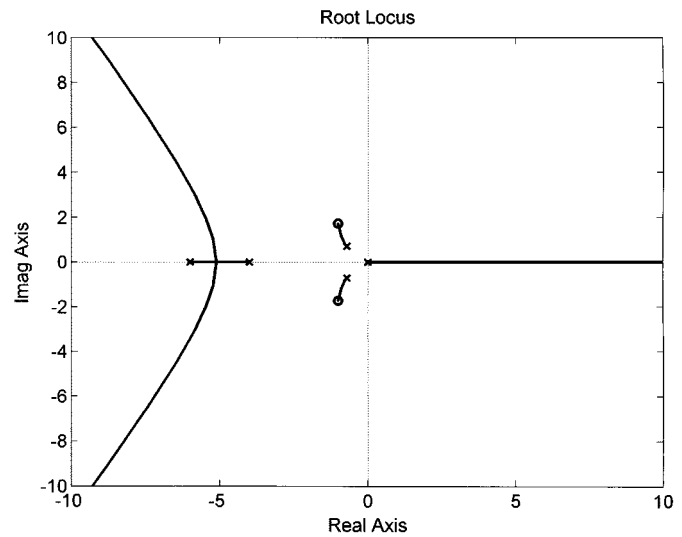Fig. 12.   Root loci plot of example 5.

Fig. 13.   Complimentary root loci plot of example 5.

To keep track of these terms, the equations used to develop this example were symbolically derived using Mathcad.

The RL plot for this system is shown in Fig. 12 and the CRL plot is shown in Fig. 13. Ogata [5] analyses the root locus plot of this system using MATLAB; MATLAB was also used to generate the root locus plots for this work.

Knowing the critical values of K, the complexity of this higher order system can be observed. As the gain increases, the system is stable until the value approaches ~ 15.6, where a region of instability is reached. The system is unstable while the gain is between 15.6 and 67.5. It is again observed to be stable while the gain is between 67.5 and 163.5, and finally it is unstable when the gain is over the 163.5 value. However, in the range where K is anywhere above 15, the system is so reactive that operation in the vicinity would not be possible; that is, the dominant poles are either in the right half plane or so close to the imaginary axis that the damping factor would be unacceptably small. Thus, the practical range for K is greater than zero, but less than 15.

The transfer function for the system defined above can be written as

$$C(s)/R(s) = K(s^2 + 2s + 4)$$
$$/(s^5 + 11.4s^4 + 39s^3 + (43.6 + K)s^2$$
$$+ (24 + 2K)s + 4K). \qquad (33)$$

Following the previous development, the real and imaginary parts of $F(\sigma + j\omega)$ are calculated, and the real and imaginary conditions are applied. From the real condition

$$\sigma^5 + 11.4\sigma^4 + 39\sigma^3 + 43.6\sigma^2 + 24\sigma$$
$$+ \omega^2(5\sigma\omega^2 - 117\sigma - 68.4\sigma^2 - 10\sigma^3 - 43.6 + 11.4\omega^2)$$
$$+ K(\sigma^2 - \omega^2 + 2\sigma + 4) = 0. \qquad (34)$$

Applying the imaginary condition another equation is derived as

$$\omega[5\sigma^4 + 45.6\sigma^3 + 117\sigma^2 + 87.2\sigma + 24$$
$$+ \omega^2(\omega^2 - 39 - 45.6\sigma - 10\sigma^2) + K(2 + 2\sigma)] = 0. \quad (35)$$

It can be observed from the above equation, the expected result, that $\omega = 0$, or the real axis, is a solution for the entire root locus.

The breakaway points on the real axis may be found from (34), and the enclosed portion of (35) with $\omega = 0$, i.e., from (34)

$$\sigma^5 + 11.4\sigma^4 + 39\sigma^3 + 43.6\sigma^2 + 24\sigma + K(\sigma^2 + 2\sigma + 4) = 0 \quad (36)$$

and from (35)

$$5\sigma^4 + 45.6\sigma^3 + 117\sigma^2 + 87.2\sigma + 24 + K(2 + 2\sigma) = 0. \quad (37)$$

Solution of the above two equations will yield the breakaway points, σ_{BP}, and the gain, K_{BP}, required to place the system poles at those breakaway points. Equation (36) may be written as

$$-K = (\sigma^5 + 11.4\sigma^4 + 39\sigma^3 + 43.6\sigma^2 + 24\sigma)/(\sigma^2 + 2\sigma + 4). \qquad (38)$$

Likewise, (37) may be solved for $-K$, i.e.,

$$-K = (5\sigma^4 + 45.6\sigma^3 + 117\sigma^2 + 87.2\sigma + 24)/(2 + 2\sigma). \quad (39)$$

Eliminating $-K$ from (38) and (39) results in a single polynomial equation.

The roots of this equation will render the breakaway points of the root locus plot from the real axis. The result of this elimination is shown below as (40)

$$3\sigma^6 + 30.8\sigma^5 + 127.4\sigma^4 + 338.4\sigma^3 + 531.2\sigma^2 + 348.8\sigma + 96 = 0 \qquad (40)$$

with roots at -5.11079, $-0.9001 \pm j2.5589$, -2.35567, and $-0.5000 \pm j0.3335$. The real roots show the location of the breakaway points. If -2.35567 is identified as σ_{BP1}, then the system gain necessary to place roots at that location, $K_{BP1} = 9.48678$, can be calculated from Equations (38) or (39). Likewise if -5.11079 is identified as σ_{BP2}, the related gain value is $K_{BP2} = -5.06492$. The first set of values is encountered

in the development of the standard root locus, and the second is associated with the complementary root locus.

After the breakaway points have been calculated, the next task is to calculate the critical values affecting stability. The gain, K_C, necessary to drive the system poles to the imaginary axis is an important parameter as well as the critical frequency, ω_C. Naturally, the real part of the pole location is zero at these points. The first step is to force omega to the point of instability by setting sigma equal to zero in the defining equations. From (34) with $\sigma = 0$, (41) is derived as

$$11.4\omega^4 - 43.6\omega^2 + K(4 - \omega^2) = 0. \qquad (41)$$

Likewise (42) is derived from (35) with $\sigma = 0$, where

$$\omega^4 - 39\omega^2 + 24 + 2K = 0. \qquad (42)$$

The above two equations may be combined to eliminate K. This is a straightforward algebraic manipulation which results in

$$\omega^6 - 20.2\omega^4 + 92.8\omega^2 - 96 = 0. \qquad (43)$$

The roots of (43) are $\pm j1.21303$, $\pm j2.15090$, and $\pm j3.75528$, which identify the critical frequencies that are labeled ω_{C1}, ω_{C2}, and ω_{C3}, respectively. Given the critical frequencies, the associated gains can be calculated from (41) or (42); the result of this calculation shows that $K_{C1} = 15.61062$, $K_{C2} = 67.51260$, and $K_{C3} = 163.55677$. The values for these gains can be verified using Routh's rule, and the values of the breakaway points can be found using a derivative of the characteristic equation. The more traditional ways require at least as much effort to apply as using Dickson's technique, and they lack the precision available for placing the system operating point.

IV. CONCLUSION

The Dickson technique is useful in allowing the designer to determine the gain required to place the systems poles and zeros at some desired location. The breakaway points and the value of gain at each point can be found by finding the conditions that exist when $\omega = 0$. The critical values of gain and frequency, for stability, can also be found by evaluating the equations at $\sigma = 0$.

For higher order systems, the derived equations become more complicated and more difficult to visualize. However, the equations are still valid, especially if two equations are derived (one that is a function of sigma and omega, and one that is a function of the gain as well as sigma or omega or both). They are intended to be used in conjunction with a computer program that generates root locus plots and can be valuable tools to increase understanding the root loci of a system.

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